Better Together: Multi-Qubit Operations

Fahimeh Bayeh

August 2025

Conventional Conditions

Why Multi-Qubit Operations?

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Quantum Conditionals

Multi-Qubit ZX-Calculus

ZX-Calculus and Multi-Qubit Gates

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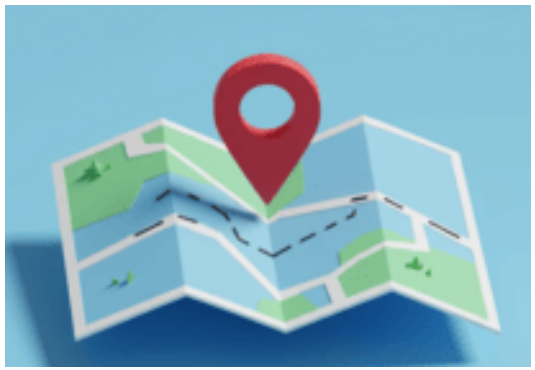

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(b) Agent-based modeling with only a single agent with one single decision

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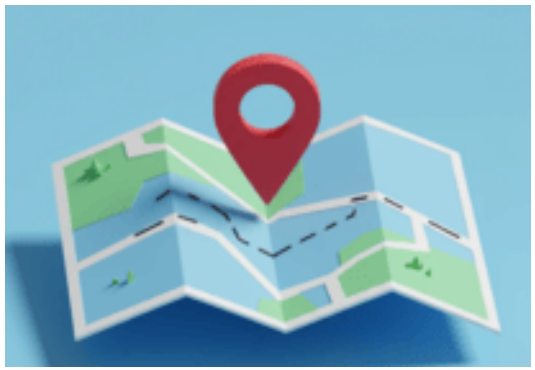

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To do more, we will need to learn about multi-qubit operations!

Conditional Statements

As we say in the last lecture, conditions are important in programming. We want to understand programs that do different things depending on whether a condition is **true** or **false**.

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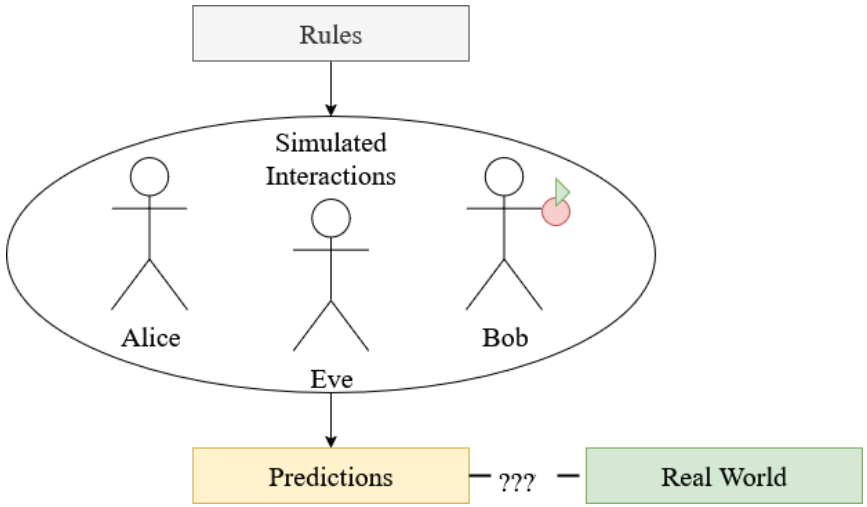
A Sociological Example

Sometimes sociologists come up with rules for how people interact, which they then want to test out on a computer.



Example: Conditional Statements in Conflicts

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Alice and Bob are not getting along, and Bob just insulted Alice! What will Alice do...?

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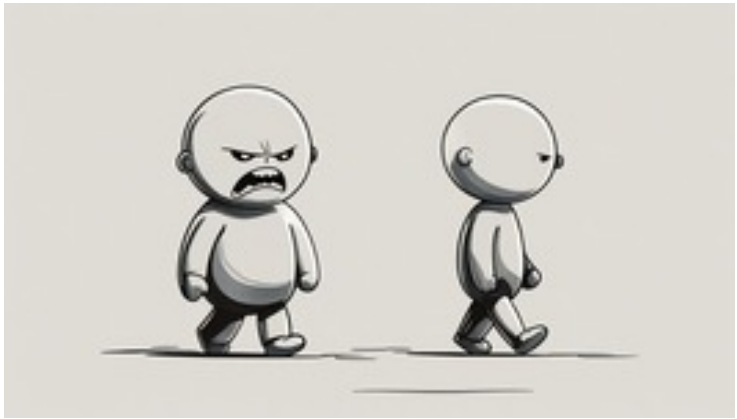

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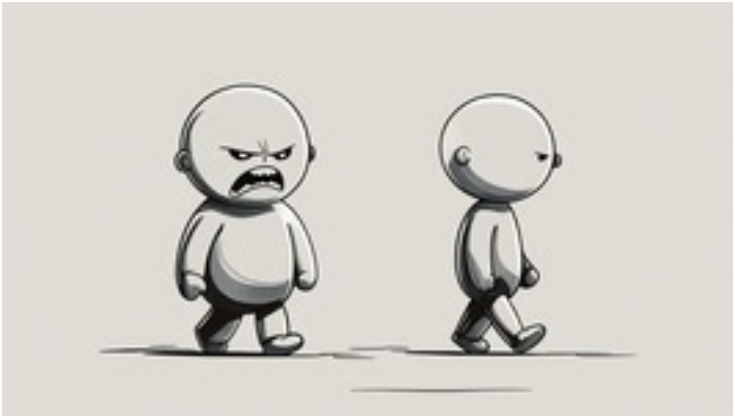

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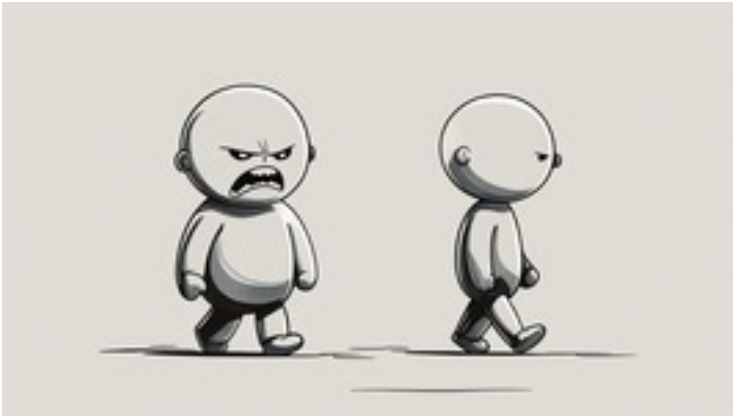

Conditional Statements as Multi-Qubit Operations

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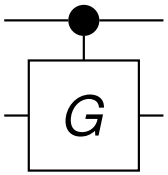
if $q = |1\rangle$ **then** $G \Leftrightarrow$ 

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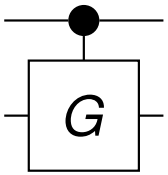

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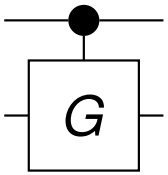

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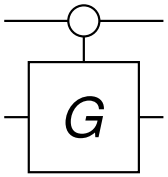

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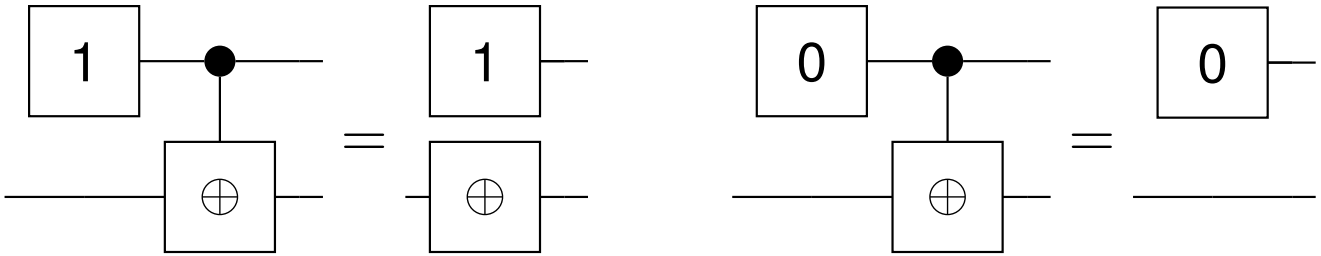

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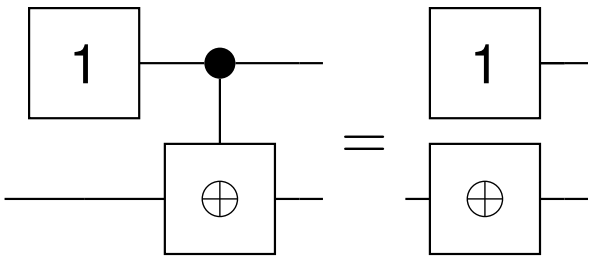

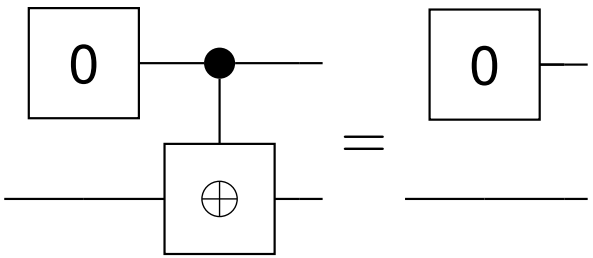

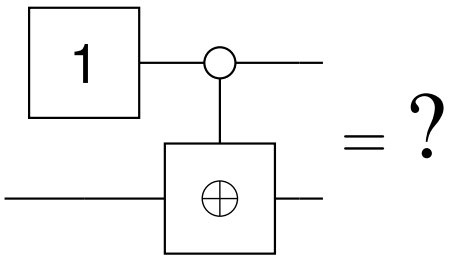

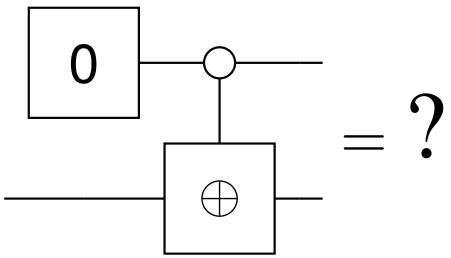

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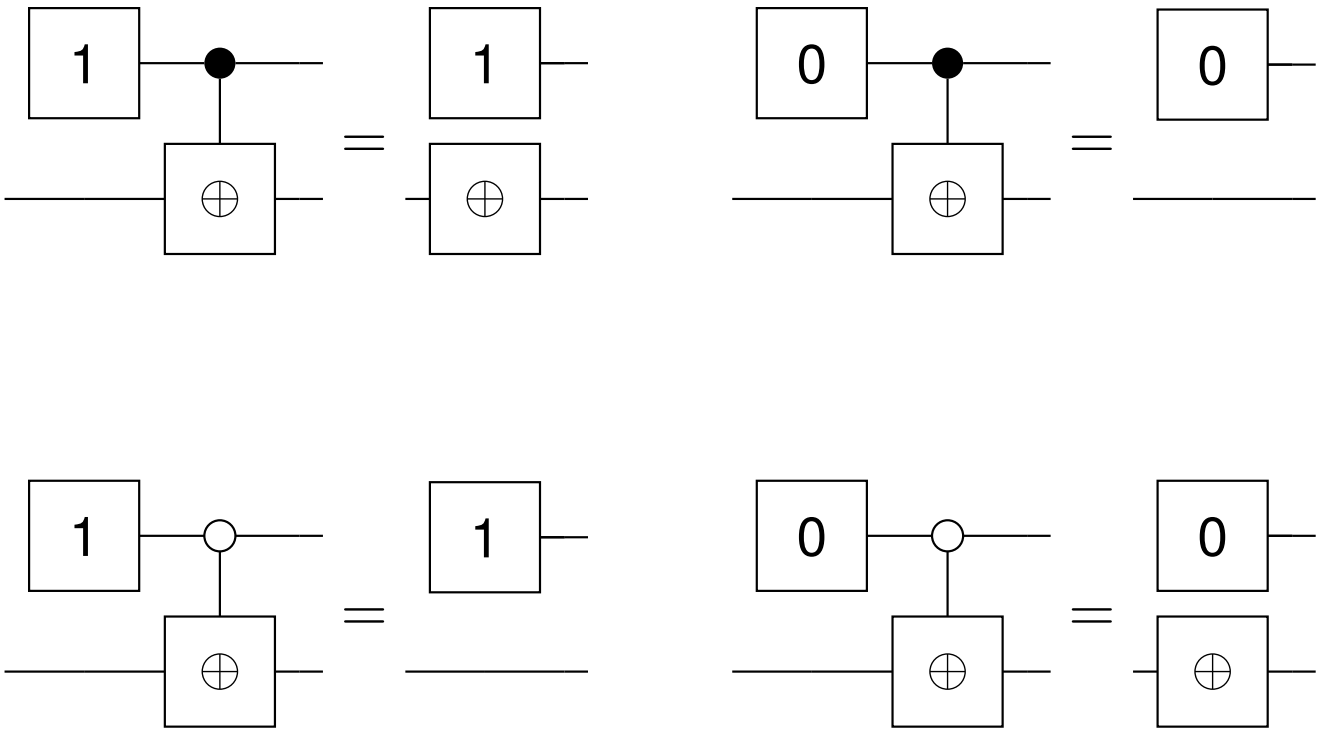
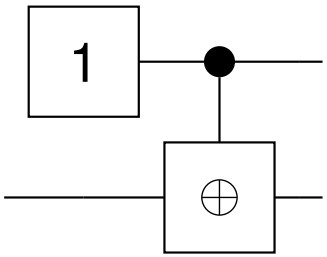
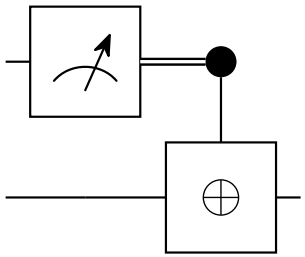


Quantum Conditionals

Controlled Gates vs. Measurement-based Control





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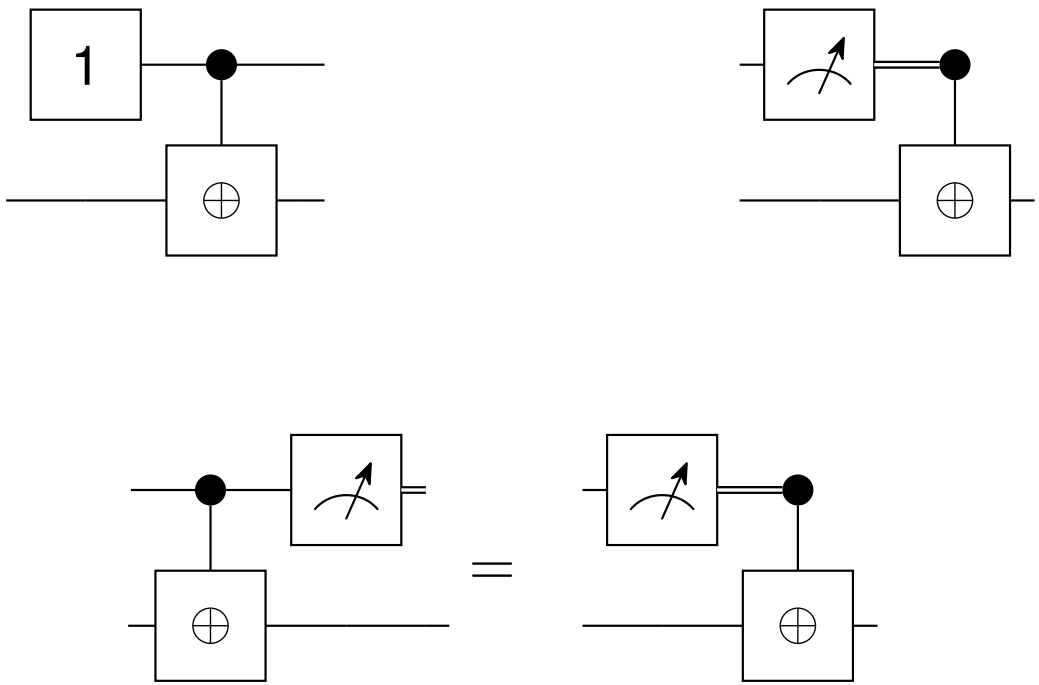

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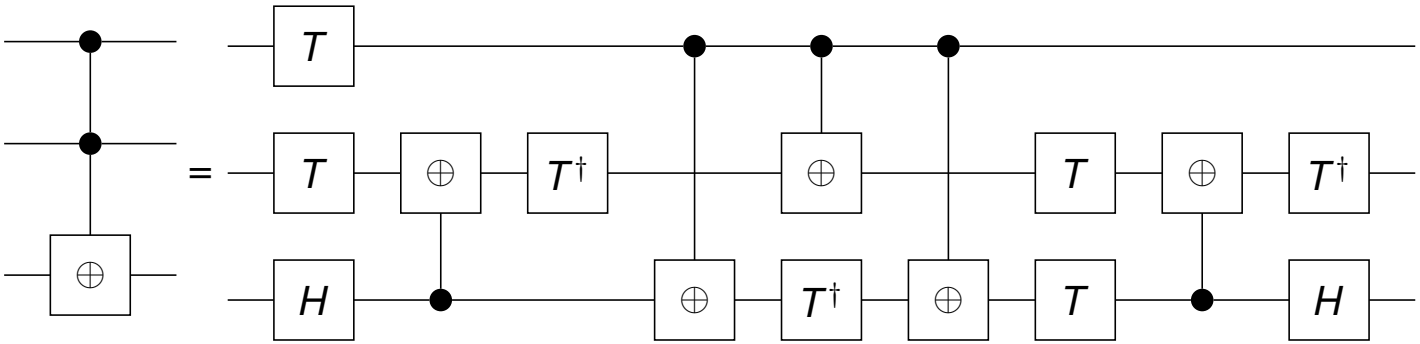
However... We must perform a measurement to get an answer...
So, we only get one answer as the state collapses.

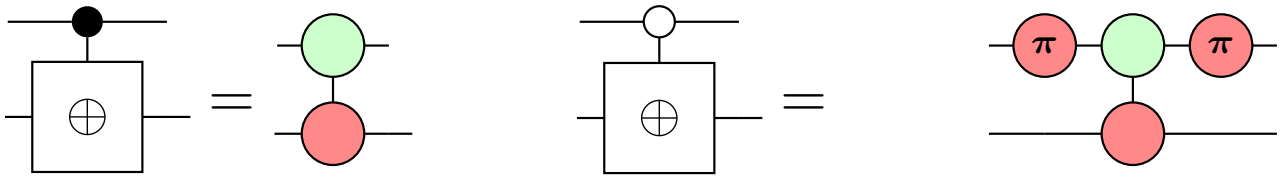
Working with Controlled NOT Gates

We could do everything with controlled NOT gates, but it is really *really* hard.



Controlled Gates in ZX-Calculus

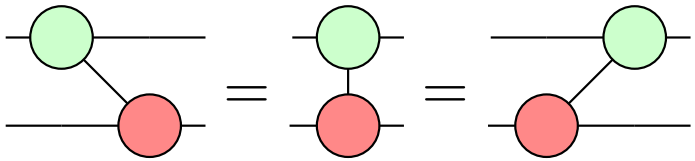
Sadly, the only controlled gates we can easily write in the ZX-calculus are the controlled NOT gates.



Multi-Qubit ZX-Calculus

Something New? Vertical Wires!

Wires in the ZX-calculus are like real wires. We can slide things along the wires as long as they don't pass through each other!



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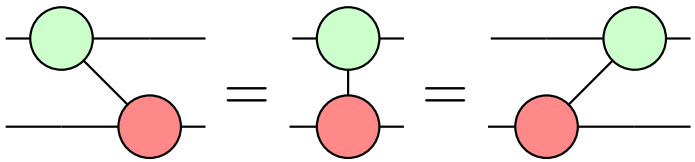
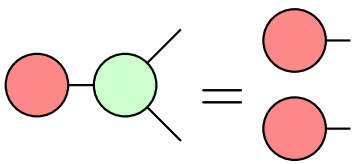


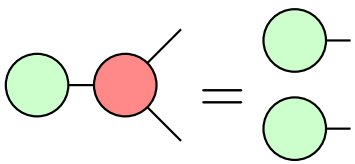
So what does this tell us about the vertical wires...? Are these dots with more than one input or output...?!

Special Dots in ZX-Calculus

Dots With One Input and Two Outputs:

- Special programs that copy states of the opposite color.





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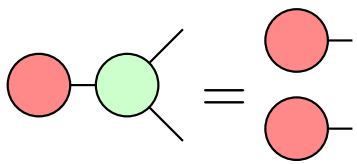

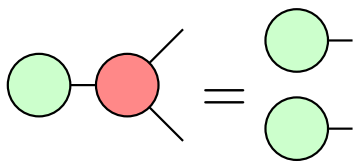

Special Dots in ZX-Calculus

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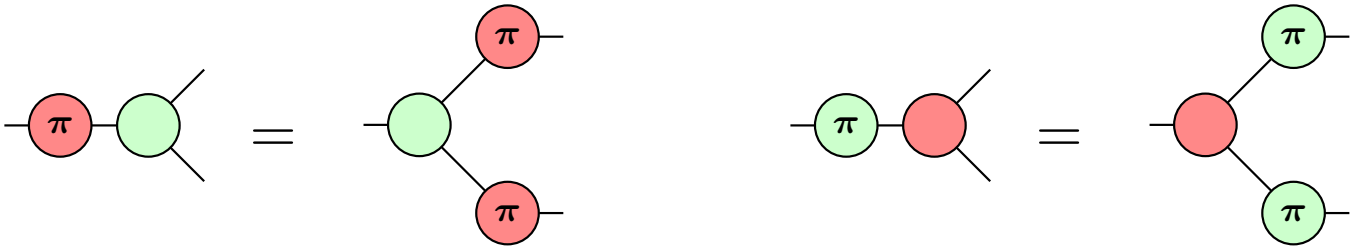
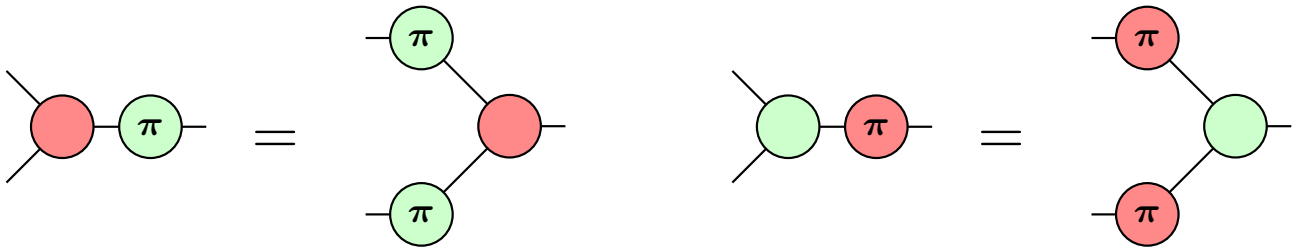
Dots With Two Inputs and One Output:

- Special programs that copy measurement outcomes of the opposite colour.

Special Dots and Negation

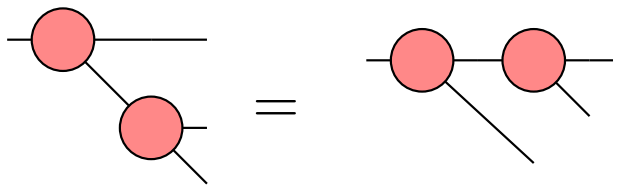
If we negate something and then copy it, then this is the same as copying and then negating both of the copies.





Properties of Copying

Copying a Second Time: It should not matter whether we copy the original or the copy.



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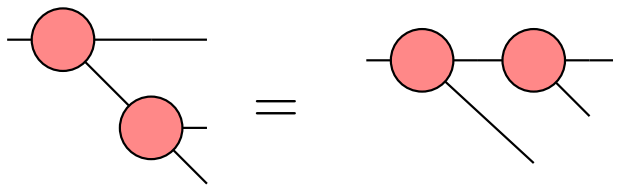
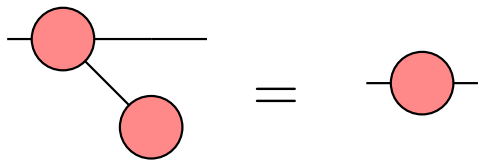


Measuring Deletes Copies: Copying and then measuring with respect to the same rotational axis is same as doing nothing.



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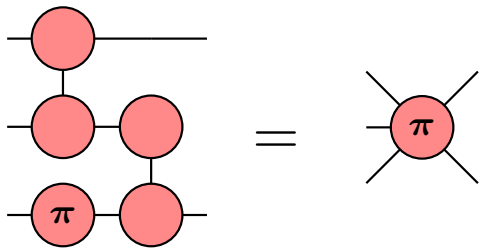

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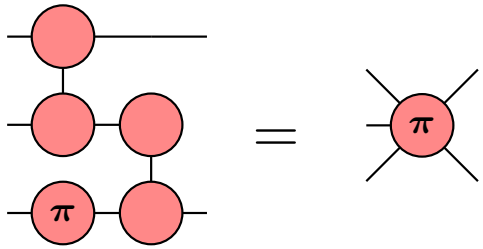
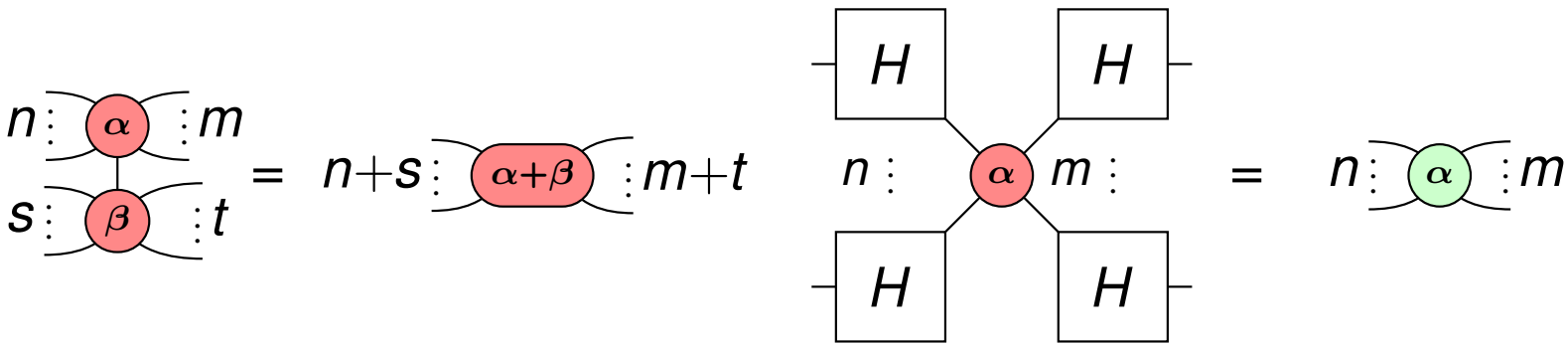
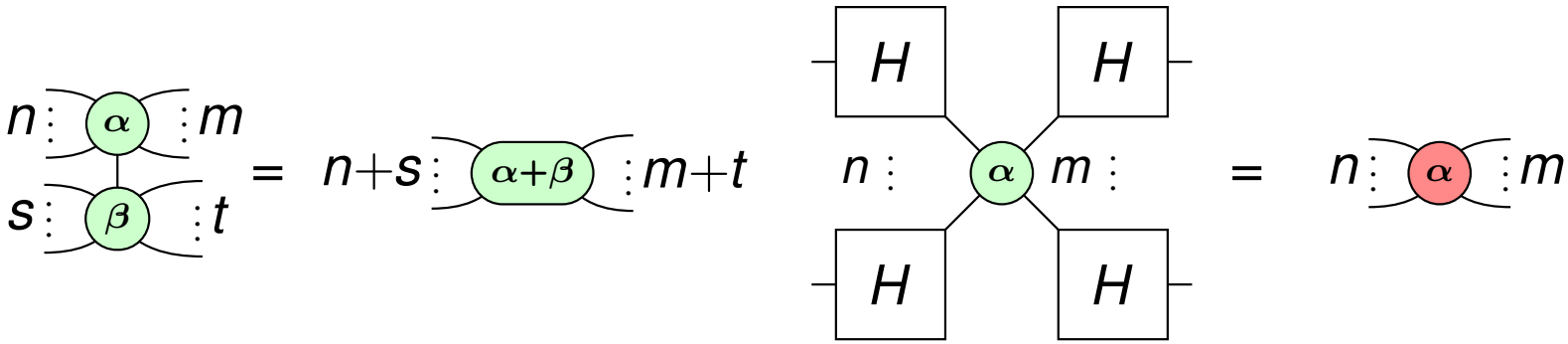


The right hand side is called a *spider*!

Rules of Spiders

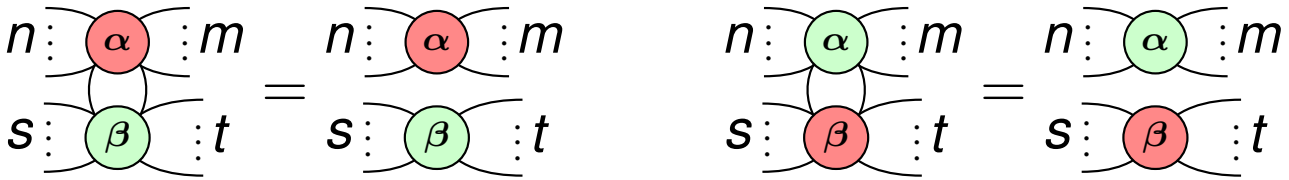




ZX-Calculus and Multi-Qubit Gates

Rules of Spiders

In the ZX-calculus, red spiders and green spiders do not get along!



Swapping Things Up

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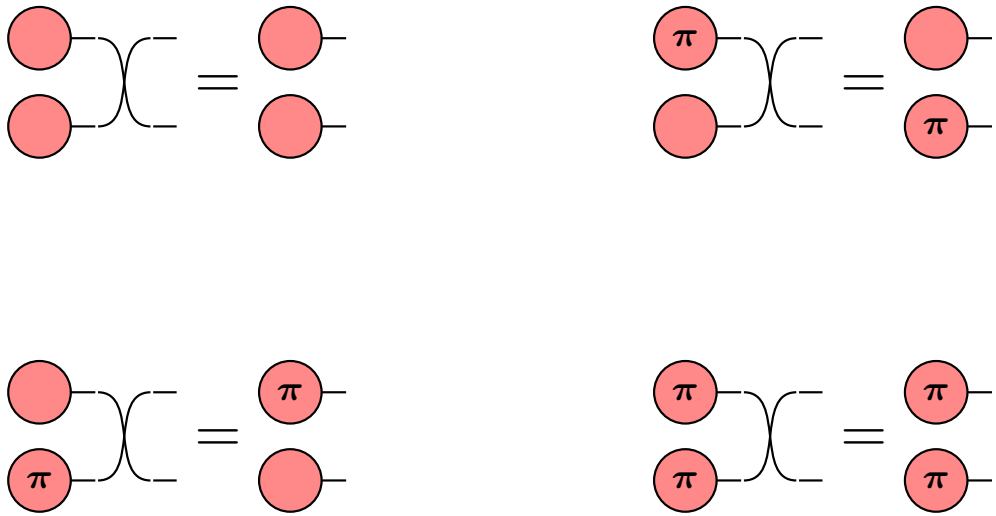
In case of qubits, swap gate works as follows:

$$|00\rangle \mapsto |00\rangle \quad |01\rangle \mapsto |10\rangle \quad |10\rangle \mapsto |01\rangle \quad |11\rangle \mapsto |11\rangle$$

Rules of Swap Gate

In a circuit diagram, we can think of the swap gate as crossing to two wires.



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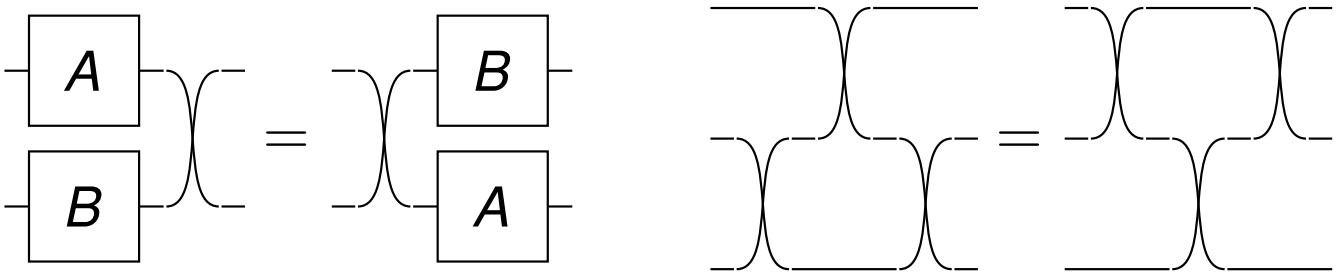
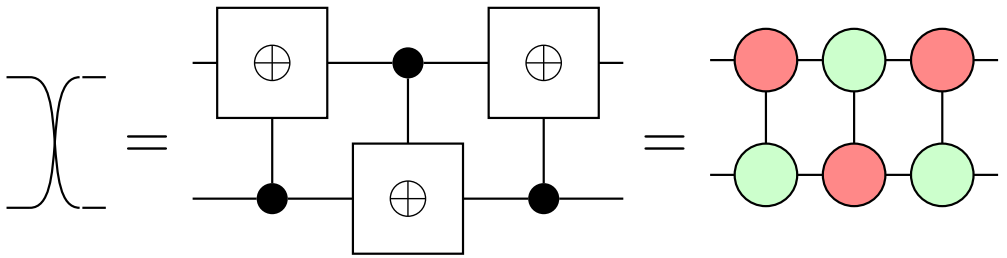


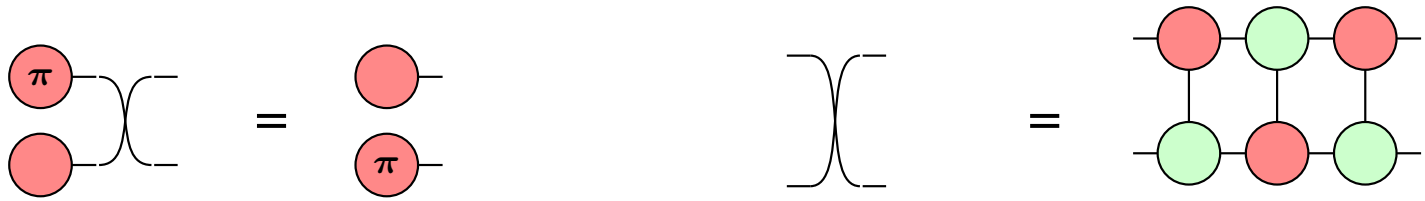
Swap Gate via Controlled-NOT

We make our own swap gate using controlled-NOT gate:

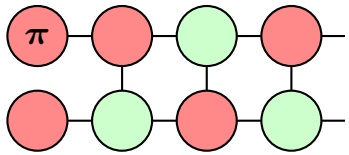


Verifying Properties of Swap Gate

We will verify the second property using properties of the controlled NOT gate:



This will take quite a few steps...



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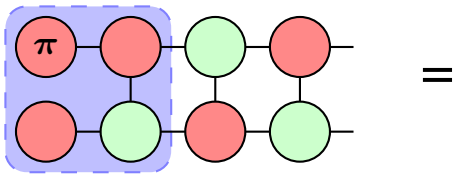

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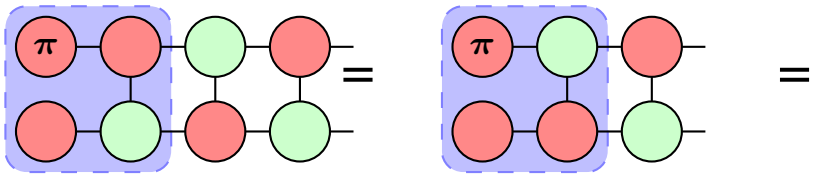

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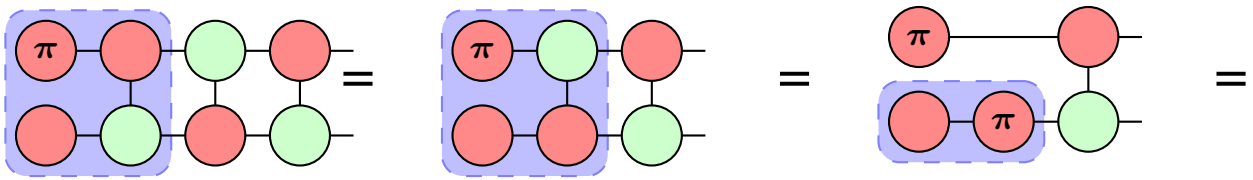

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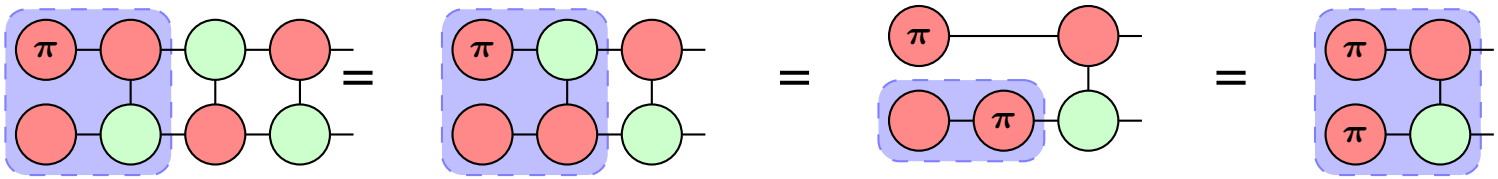

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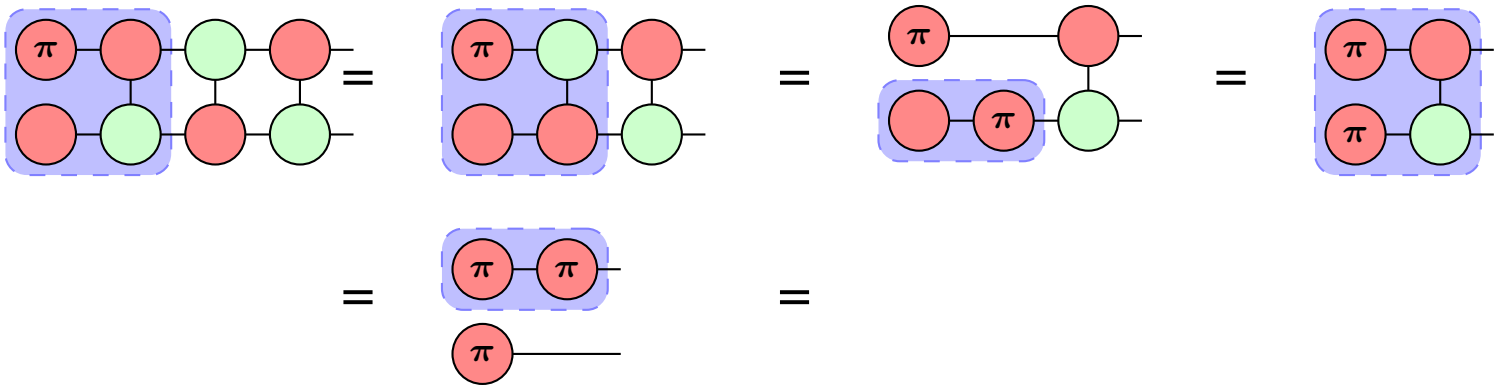

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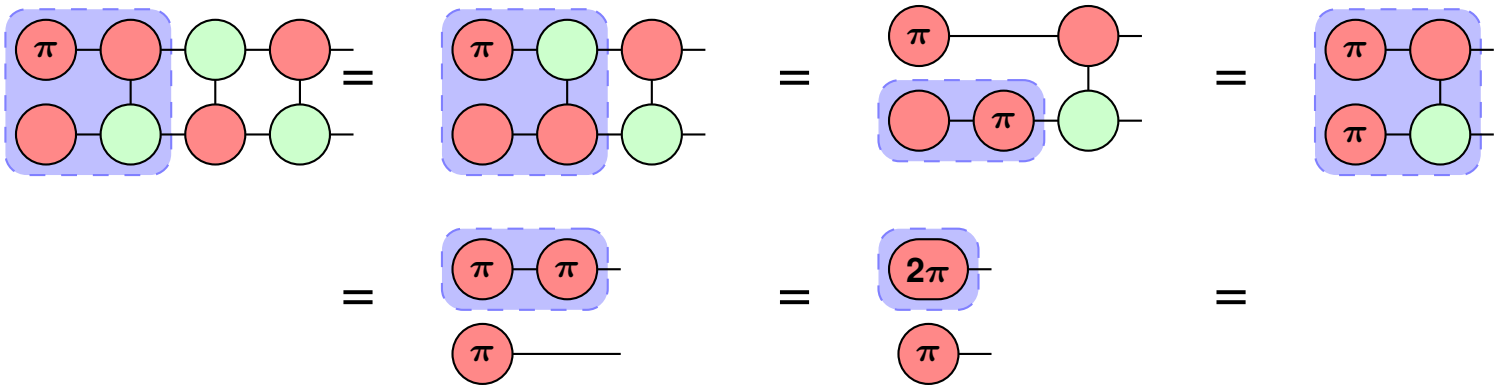

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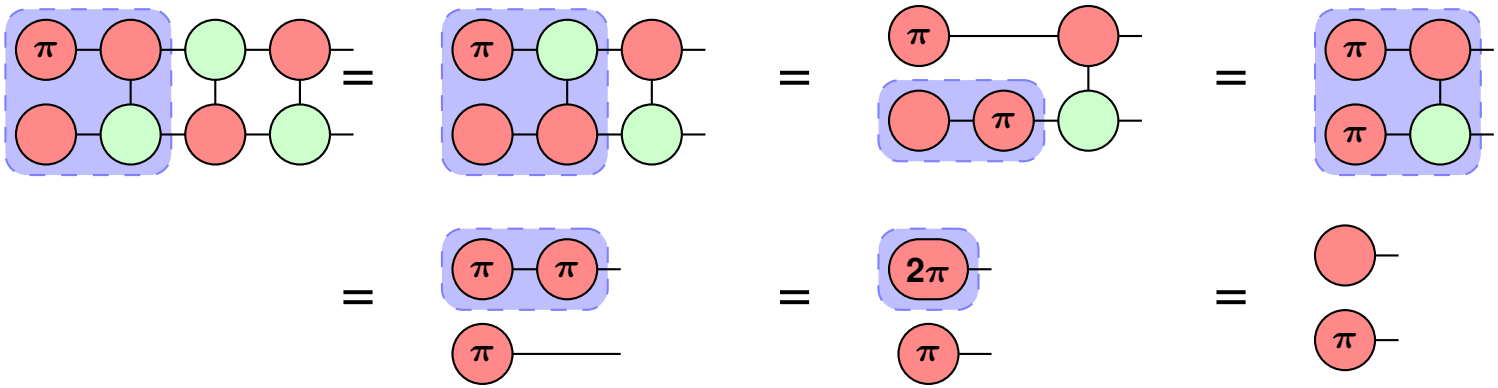

Question

Verify the fourth property as well, using properties of the controlled NOT gate:

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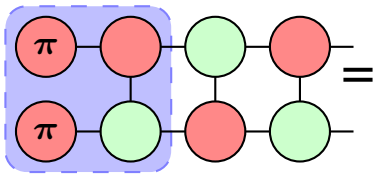

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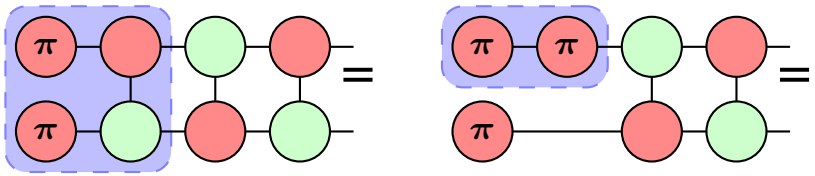

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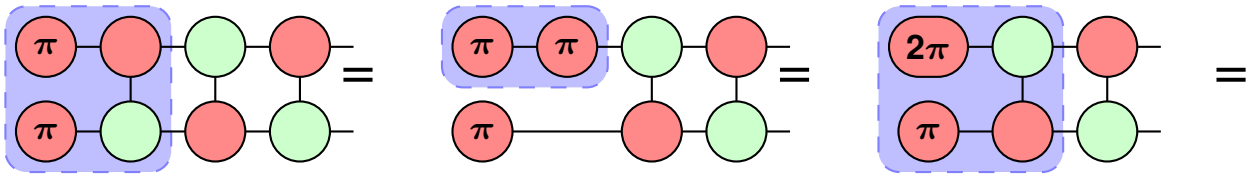

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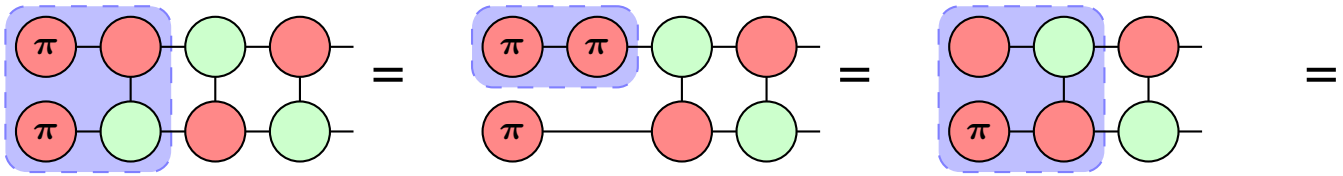

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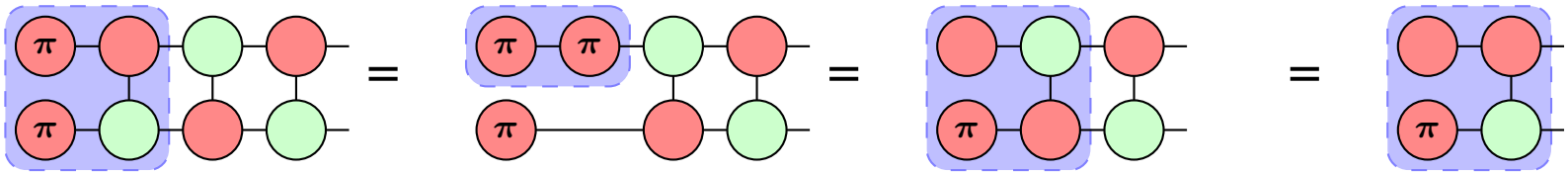

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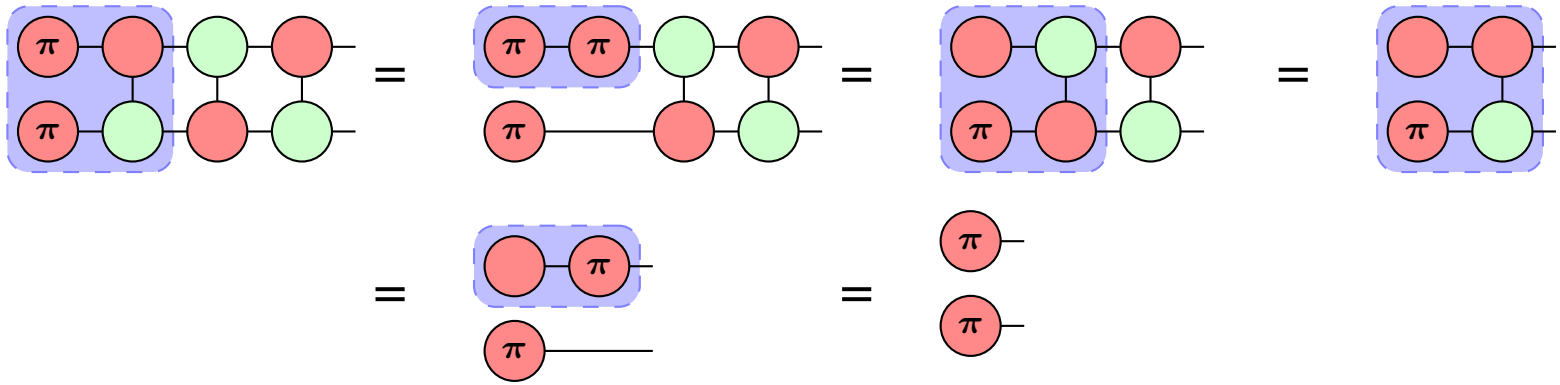

Photo Credits

The following were created specifically for this presentation.

- All circuit diagrams.
- All ZX-diagrams.
- The illustration of Alice, Bob, and Eve interacting.

All other illustrations were generated using MonshaAI.